Section 7.2 Transformation of IVP

utilize Laplace transform to solve DE w constant coefficients

fmt: $ax''(t) + bx'(t) + cx(t) = f(t)$ w initial cond: $x(0) = x_0$, $x'(0) = x_0'$

becomes $a\mathcal{L}\{x''(t)\} + b\mathcal{L}\{x'(t)\} + c\mathcal{L}\{x(t)\} = \mathcal{L}\{f(t)\}$

Transform of Derivatives

let $f(t)$ be continuous

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$
$$= s\,F(s) - f(0)$$

then $\mathcal{L}\{f''(t)\}$ let $g(t) = f'(t)$

$$= \mathcal{L}\{g'(t)\}$$
$$= s\,\mathcal{L}\{g(t)\} - g(0)$$
$$= s\,\underline{\mathcal{L}\{f'(t)\}} - f'(0)$$
$$= s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0)$$
$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\boxed{\mathcal{L}\{f''(t)\} = s^2\,F(s) - sf(0) - f'(0)}$$

similarly, $\mathcal{L}\{f'''(t)\} = s\,\underline{\mathcal{L}\{f''(t)\}} - f''(0)$
$$= s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$\therefore \boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - \ldots sf^{(n-2)}(0) - f^{(n-1)}(0)}$$

ex. solve $x'' - x' - 6x = 0$ given $x(0) = 2$ $x'(0) = -1$

$$\mathcal{L}\{x''(t)\} - \mathcal{L}\{x'(t)\} - 6\mathcal{L}\{x(t)\} = 0$$

$$s\mathcal{L}\{x(t)\} - x(0)$$
$$s\,X(s) - 2$$

$$s\mathcal{L}\{x(t)\} - x(0)$$
$$s\,X(s) - 2$$

$$\downarrow$$

$$s^2 X(s) - s\,x(0) - x'(0)$$
$$s^2 X(s) - 2s - (-1)$$

then $x'' - x' - 6x = 0$

$$[s^2X(s) - 2s + 1] - [sX(s) - 2] - 6X(s) = 0$$
$$\underline{s^2X(s)} - 2s + 1 - \underline{sX(s)} + 2 - \underline{6X(s)} = 0$$
$$(s^2 - s - 6)X(s) - 2s + 3 = 0 \qquad \text{solve for } X(s)$$
$$X(s) = \frac{2s - 3}{s^2 - s - 6} \qquad \text{use partial fractions}$$

$$\frac{2s-3}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$
$$2s - 3 = As + 2A + Bs - 3B$$
$$2 = A + B \qquad B = 2 - A$$
$$-3 = 2A - 3(2 - A)$$
$$-3 = 2A - 6 + 3A$$
$$3 = 5A \Rightarrow A = \frac{3}{5} \quad B = \frac{7}{5}$$

$$X(s) = \frac{\frac{3}{5}}{s-3} + \frac{\frac{7}{5}}{s+2}$$
$$= \frac{3}{5}\cdot\boxed{\frac{1}{s-3}} + \frac{7}{5}\cdot\boxed{\frac{1}{s+2}} \qquad \text{recall } \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\boxed{x(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}}$$

ex. solve $\underbrace{x''} + \underbrace{4x} = \sin 3t$ (k=3) given $x(0)=0$ and $x'(0)=0$

$$s^2X(s) - 0 - 0 + 4X(s) = \frac{3}{s^2+9} \qquad \text{recall: } \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$
$$X(s) = \frac{3}{(s^2+4)(s^2+9)}$$

$$= \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$3 = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$3 = As^3 + 9As + Bs^2 + 9B + Cs^3 + 4Cs + Ds^2 + 4D$$

$$0 = A + C \qquad 9A + 4C = 0 \qquad A = C = 0$$

$$B + D = 0 \Rightarrow D = -B$$

$$3 = 9B + 4D$$
$$3 = 9B - 4B$$
$$3 = 5B \Rightarrow B = \frac{3}{5}, \quad D = -\frac{3}{5}$$

$$X(s) = \frac{3}{5} \cdot \frac{1}{\underbrace{s^2+4}_{a=2}} - \frac{3}{5} \cdot \frac{1}{\underbrace{s^2+9}_{a=3}}$$

$$x(t) = \frac{3}{5} \cdot \frac{1}{2} \sin 2t - \frac{3}{5} \cdot \frac{1}{3} \sin 3t$$

$$\boxed{x(t) = \frac{3}{10} \sin 2t - \frac{1}{5} \sin 3t}$$

ex. find $\mathcal{L}\{t \sin kt\}$

let $f(t) = t \sin kt$

then $f(0) = 0$

$$f'(t) = \sin kt + tk\cos kt$$
$$f'(0) = 0 + 0 = 0$$

$$f''(t) = k\cos kt + k\cos kt - tk^2 \sin kt$$
$$f''(t) = 2k\cos kt - k^2 \underbrace{t \sin kt}$$

$$\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - 0 - 0$$
$$= s^2 \mathcal{L}\{t \sin kt\}$$

add Laplace to each term

also $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

$$s^2 \mathcal{L}\{t \sin kt\} = 2k \quad s \quad - k^2 \mathcal{L}\{t \sin kt\}$$

also $\mathcal{L}\{\cos kt\} = \overline{s^2+k^2}$

$$s^2 \mathcal{L}\{t\sin kt\} = 2k \cdot \frac{s}{s^2+k^2} - k^2 \mathcal{L}\{t\sin kt\}$$

$$(s^2+k^2)\mathcal{L}\{t\sin kt\} = \frac{2ks}{s^2+k^2}$$

$$\boxed{\mathcal{L}\{t\sin kt\} = \frac{2ks}{(s^2+k^2)^2}}$$

much easier than $\int_0^\infty e^{-st}(t\sin kt)\,dt$

Section 7.3

so far:

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$t^n \quad n \geq 0$	$\frac{n!}{s^{n+1}}$
t	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sin kt$	$= \frac{k}{s^2+k^2}$

sometimes, it's useful to $\mathcal{L}^{-1}\{F(s)\}$ by writing it as a rational function:

$$R(s) = \frac{Q(s)}{P(s)}$$ where degree $P(s) <$ degree $Q(s)$

will explain why next time

then can use similar rules to partial fractions

know $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ if $F(s) = \mathcal{L}\{f(t)\}$ exists

then $\mathcal{L}\{e^{at}f(t)\}$ exists and

$= F(s-a)$

$$= F(s-a)$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

add to list: $e^{at} t^n = \dfrac{n!}{(s-a)^{n+1}}$ $s > a$

$$e^{at} \cos kt = \frac{s-a}{(s-a)^2 + k^2}$$

$$e^{at} \sin kt = \frac{k}{(s-a)^2 + k^2}$$

Formulas

$$\mathcal{L}\{x''(t)\} = s^2 X(s) - s\,x(0) - x'(0)$$

$$\mathcal{L}\{x'(t)\} = s X(s) - x(0)$$

ex. solve DE $\frac{1}{2}x'' + 3x' + 17x = 0$ given $x(0) = 3$ and $x'(0) = 1$

$$2\left[\tfrac{1}{2}\left(s^2 X(s) - 3s - 1\right) + 3\left(s X(s) - 3\right) + 17 X(s) = 0\right]$$

$$s^2 X(s) - 3s - 1 + 6s X(s) - 18 + 34 X(s) = 0$$

$$\left(s^2 + 6s + 34\right) X(s) = 3s + 19$$

$$X(s) = \frac{3s+19}{s^2 + 6s + 34} \leftarrow \text{complete square}$$

$s^2 + 6s + 9 = -34 + 9$

$(s+3)^2 + 25$

$$= \frac{3s+19}{\underbrace{(s+3)^2}_{s^2} + \underbrace{25}_{k^2}}$$

$$3s+19 = \cancel{A}^{3} \cdot \frac{s+3}{(s+3)^2+25} + \cancel{B}^{2}\, \frac{5}{(s+3)^2+25}$$

$s-a$ (→ $s+3$), k (→ 5)

$$3s + 19 = As + 3A + 5B$$

$$A = 3$$

$$19 = 9 + 5B$$

$$10 = 5B \quad = B = 2$$

take $\mathcal{L}^{-1}$ to get $\boxed{x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t}$

take $\mathcal{L}^{-1}$ to get $\boxed{x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t}$